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Note

Dynamical Interactions of Superconducting Flux Vortices*

We perform a numerical simulation of the dynamical interactions of magnetic flux tubes ("vortices") in a superconductor. The system is described in the Ginzburg–Landau theory by the action

$$S=\int d^4x \left\{-(\nabla^\mu\varphi)^\dagger(\nabla_\mu\varphi)-\frac{\lambda}{4}(\varphi^\dagger\varphi-\sigma^2)^2-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right\}. \tag{1}$$

Here $\varphi(x)$ is a complex scalar field representing the Fermi gap parameter of the superconductor, $F_{\mu\nu}$ is the electromagnetic field strength tensor,

$$F_{\mu\nu}=\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x)$$
$$=\begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \tag{2}$$

and the gauge-covariant derivative is

$$\nabla_\mu\varphi(x)=\partial_\mu\varphi(x)-ieA_\mu(x)\,\varphi(x). \tag{3}$$

The potential for the $\varphi(x)$ field is the Higgs potential (or "Mexican hat" potential) shown in Fig. 1. The vacuum (lowest energy state) of this system is degenerate and consists of the set of points $\varphi(x)=\sigma e^{i\theta_0}$ for any fixed θ_0. Since the vacuum is topologically non-trivial it is possible to create topological defects (called "vortices"), where the $\varphi(x)$ field "winds" around the central maximum of the potential like

$$\varphi(r,\theta)=f(r)\,e^{in\theta}. \tag{4}$$

Because $\varphi(x)$ must be continuous n must be an integer and $\varphi(x)$ must vanish at the center of the vortex. The phase and magnitude of $\varphi(x)$ near an $n=1$ vortex are shown in Fig. 2. Outside of the vortex the electromagnetic gauge symmetry is

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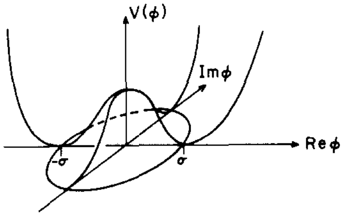

FIG. 1. Higgs potential for the scalar field $\varphi(x)$.

broken and the material is superconducting. Inside the vortex the gauge symmetry is restored and there is a magnetic field in the z direction. The total magnetic flux through the vortex is quantized and is given by

$$\Phi(B) = 2\pi n/e. \tag{5}$$

The vortices therefore represent quantized tubes of magnetic flux trapped in a superconducting medium.

From a previous variational calculation of the interaction energy of stationary vortices [1] it was known that for the scalar field coupling constant $\lambda < 2$ (corresponding to a Type I superconductor) vortices attract each other, while for $\lambda > 2$ (corresponding to a Type II superconductor) they repel. At the critical coupling $\lambda = 2$ it was shown that *isolated* vortices do not interact, but it was not known if critically coupled vortices when brought together would interact or would simply

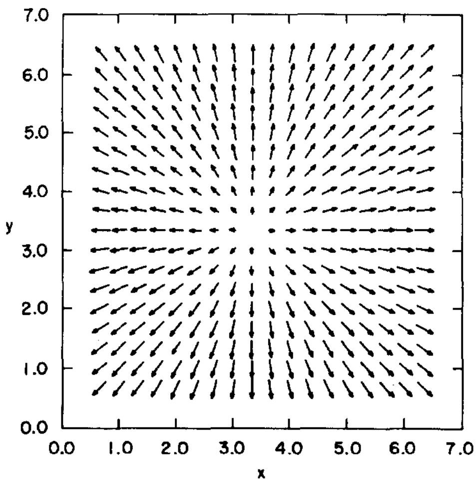

FIG. 2. Phase and magnitude of $\varphi(x)$ near a vortex.

pass through each other as solitons. Our simulation shows that critically coupled vortices do in fact interact non-trivially and are therefore not solitons.

To simulate the system on a computer we discretize the action using techniques from lattice gauge field theory [2]. The field $\varphi(x)$ is represented by the variables $\varphi_{\mathbf{x}}$, which live on the sites of the lattice, while the gauge fields are represented by the variables $\theta^{\mu}_{\mathbf{x}}$, which live on the links of the lattice. To covariantly transport $\varphi_{\mathbf{x}}$ across a link in the forward direction one must multiply by $\exp(i\theta^{\mu}_{\mathbf{x}})$, so the lattice version of the covariant derivative is

$$\nabla_{\mu}\varphi_{\mathbf{x}}=\frac{1}{a}\left[\exp(-i\theta^{\mu}_{\mathbf{x}})\,\varphi_{\mathbf{x}+\boldsymbol{\mu}}-\varphi_{\mathbf{x}}\right]. \tag{6}$$

With $\pi^{\dagger}_{\mathbf{x}}$ and $E^{\mu}_{\mathbf{x}}$ the momenta conjugate to the fields $\varphi_{\mathbf{x}}$ and $\theta^{\mu}_{\mathbf{x}}$ and $A_{\mathbf{x}}$ representing $A_0(x)$ the Hamiltonian for the discretized system is

$$\begin{aligned}H=\sum_{\mathbf{x}} a^{d}\Bigg\{&\pi_{\mathbf{x}}(\tfrac{1}{2}\pi^{\dagger}_{\mathbf{x}}-ieA_{\mathbf{x}}\varphi^{\dagger}_{\mathbf{x}})+\pi^{\dagger}_{\mathbf{x}}(\tfrac{1}{2}\pi_{\mathbf{x}}+ieA_{\mathbf{x}}\varphi^{\dagger}_{\mathbf{x}})\\
&+\tfrac{1}{2}\sum_{i}(E^{i}_{\mathbf{x}})^{2}+\sum_{i}(\nabla_{i}\varphi_{\mathbf{x}})^{\dagger}(\nabla_{i}\varphi_{\mathbf{x}})+\tfrac{1}{4}(\varphi^{\dagger}_{\mathbf{x}}\varphi_{\mathbf{x}}-\sigma^{2})^{2}\\
&+\frac{1}{2a^{4}}\sum_{i\neq j}\left[1-\cos(\theta^{i}_{\mathbf{x}}+\theta^{j}_{\mathbf{x}+\mathbf{i}}-\theta^{i}_{\mathbf{x}+\mathbf{j}}-\theta^{j}_{\mathbf{x}})\right]\\
&+\frac{1}{a}\sum_{i}E^{i}_{\mathbf{x}}(A_{\mathbf{x}+\mathbf{i}}-A_{\mathbf{x}})\Bigg\}.\end{aligned} \tag{7}$$

The virtue of using the lattice gauge field theory formalism is that it preserves the local gauge symmetry of the system. The Hamiltonian above is invariant under the transformation:

$$\varphi_{\mathbf{x}}\to\exp(i\chi_{\mathbf{x}})\,\varphi_{\mathbf{x}} \tag{8a}$$

$$\pi_{\mathbf{x}}\to\exp(i\chi_{\mathbf{x}})\,\pi_{\mathbf{x}} \tag{8b}$$

$$\theta^{\mu}_{\mathbf{x}}\to\theta^{\mu}_{\mathbf{x}}+\chi_{\mathbf{x}+\boldsymbol{\mu}}-\chi_{\mathbf{x}} \tag{8c}$$

$$E^{\mu}_{\mathbf{x}}\to E^{\mu}_{\mathbf{x}} \tag{8d}$$

with $\chi_{\mathbf{x}}$ an arbitrary function of position. As a consequence of this symmetry the system obeys the constraint

$$\sum_{i}\frac{1}{a}(E^{i}_{\mathbf{x}+\mathbf{i}}-E_{\mathbf{x}})=i(\pi_{\mathbf{x}}\varphi^{\dagger}_{\mathbf{x}}-\pi^{\dagger}_{\mathbf{x}}\varphi_{\mathbf{x}}) \tag{9}$$

which is the lattice version of Gauss' law.

The dynamics of the system are described by Hamilton's equations for H in Eq. (7). The equations of motion are

$$\frac{d\varphi_{\mathbf{x}}}{dt} = \pi_{\mathbf{x}} \tag{10a}$$

$$\frac{d\theta^i_{\mathbf{x}}}{dt} = aE^i_{\mathbf{x}} \tag{10b}$$

$$\begin{aligned}\frac{d\pi_{\mathbf{x}}}{dt} = {} & \frac{1}{a^2} \sum_i \left[\exp(i\theta^i_{\mathbf{x}-\mathbf{i}})\, \varphi_{\mathbf{x}-\mathbf{i}} + \exp(-i\theta^i_{\mathbf{x}})\, \varphi_{\mathbf{x}+\mathbf{i}}\right] - \frac{2d}{a^2} \varphi_{\mathbf{x}} \\ & - \frac{\lambda}{2} (\varphi^\dagger_{\mathbf{x}} \varphi_{\mathbf{x}} - \sigma^2)\, \varphi_{\mathbf{x}}\end{aligned} \tag{10c}$$

$$\begin{aligned}\frac{dE^i_{\mathbf{x}}}{dt} = {} & -\frac{ie}{a} \left[\varphi^\dagger_{\mathbf{x}} \exp(-i\theta^i_{\mathbf{x}})\, \varphi_{\mathbf{x}+\mathbf{i}} - \varphi^\dagger_{\mathbf{x}+\mathbf{i}} \exp(i\theta^i_{\mathbf{x}})\, \varphi_{\mathbf{x}}\right] \\ & - \frac{e}{a^3} \sum_{j \neq i} \{\sin(\theta^i_{\mathbf{x}} + \theta^j_{\mathbf{x}+\mathbf{i}} - \theta^i_{\mathbf{x}+\mathbf{j}} - \theta^j_{\mathbf{x}}) - \sin(\theta^i_{\mathbf{x}-\mathbf{j}} + \theta^j_{\mathbf{x}+\mathbf{i}-\mathbf{j}} - \theta^i_{\mathbf{x}} - \theta^j_{\mathbf{x}-\mathbf{j}})\}.\end{aligned} \tag{10d}$$

In obtaining these equations we have used the gauge symmetry of the system to impose the condition $A_{\mathbf{x}} = 0$. This both simplifies the calculations and makes the Hamiltonian in Eq. (7) manifestly positive-definite.

Figures 3 through 11 show the results of the simulation of a head-on collision between two critically coupled vortices with $\lambda = 2$ and $a = 0.33$. We plot the total energy density of the system as a function of $x-y$ position at succesive times. The initial configuration is such that the vortices approach each other in the x direction at 0.5 times the speed of light. As the figures show the two vortices come together and then separate at right angles, clearly demonstrating that there is a non-trivial interaction between them. When two vortices are put in the same positions but without any initial velocity they remain where they are and neither attract nor repel.

The same set of equations in three dimensions describes the dynamics of "cosmic strings," which are thin tubes of false vacuum that may have formed during a phase transition in the early cosmological evolution of the universe [3]. It is an important open question as to whether crossing cosmic strings simply pass through each other or whether they "intercommute" (i.e., trade ends). If, as is often assumed, cosmic strings intercommute then they can form loops. These loops would be density perturbations in the early universe around which matter would accrete, leading to the formation of galaxies and clusters of galaxies. An interesting property of these strings is that they are scale-invariant, so loops of all sizes would form, leading to a scale-invariant distribution of galaxies and clusters of galaxies, which matches observation. If cosmic strings do not intercommute then there is no presently known explaination for the observed scaling distributions of galaxies. We intend to use our codes to simulate the interactions of cosmic strings to determine whether or not they intercommute.

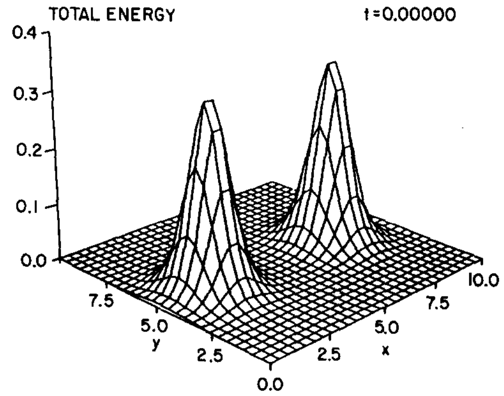

FIG. 3. Energy density of two colliding vortices; $t = 0.00$.

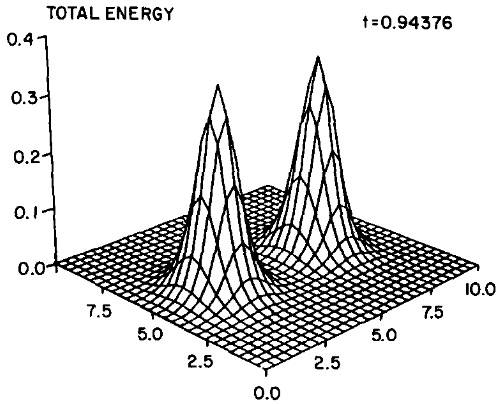

FIG. 4. Energy density of two colliding vortices; $t = 0.94$.

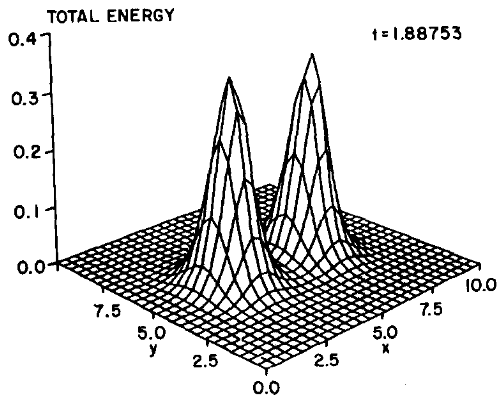

FIG. 5. Energy density of two colliding vortices; $t = 1.89$.

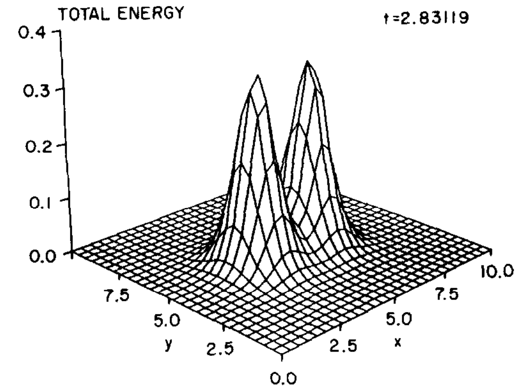

FIG. 6. Energy density of two colliding vortices; $t = 2.83$.

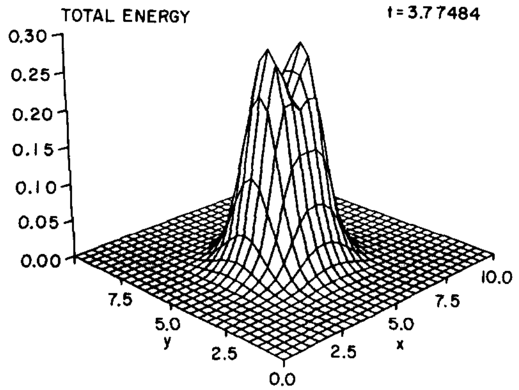

FIG. 7. Energy density of two colliding vortices; $t = 3.77$.

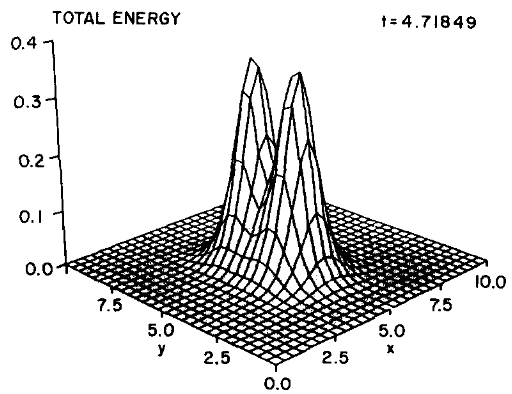

FIG. 8. Energy density of two colliding vortices; $t = 4.72$.

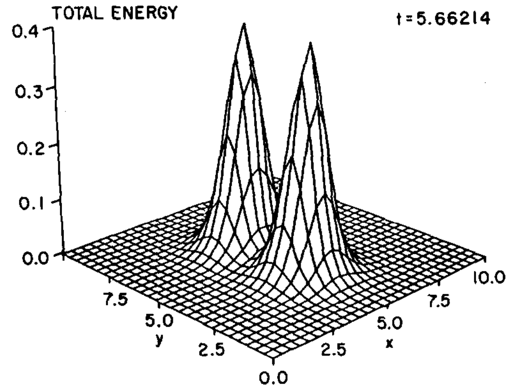

FIG. 9. Energy density of two colliding vortices; $t = 5.66$.

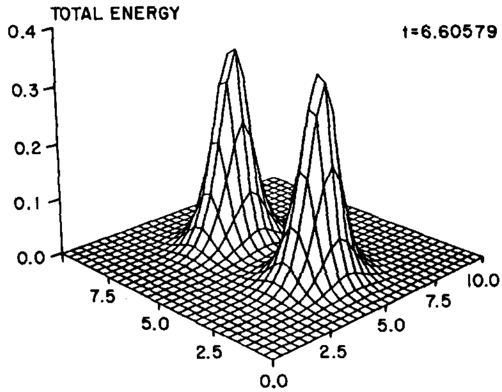

FIG. 10. Energy density of two colliding vortices; $t = 6.61$.

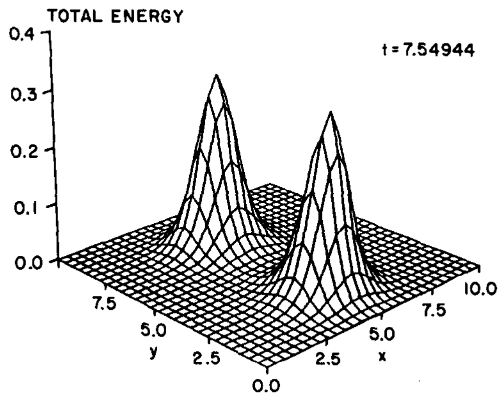

FIG. 11. Energy density of two colliding vortices; $t = 7.55$.

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